
Jackson Electrostatics - I 09-16-17

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Initialization: Be sure the files *NTGStylesheet2.nb* and *NTGUtilityFunctions.m* is are in the same directory as that from which this notebook was loaded. Then execute the cell immediately below by mousing left on the cell bar to the right of that cell and then typing “shift” + “enter”. Respond “Yes” in response to the query to evaluate initialization cells.

In[15]:=

```
SetDirectory[NotebookDirectory[]];
(* set directory where source files are located *)
SetOptions[EvaluationNotebook[], (* load the StyleSheet *)
  StyleDefinitions -> Get["NTGStylesheet2.nb"]];
Get["NTGUtilityFunctions.m"]; (* Load utilities package *)
```

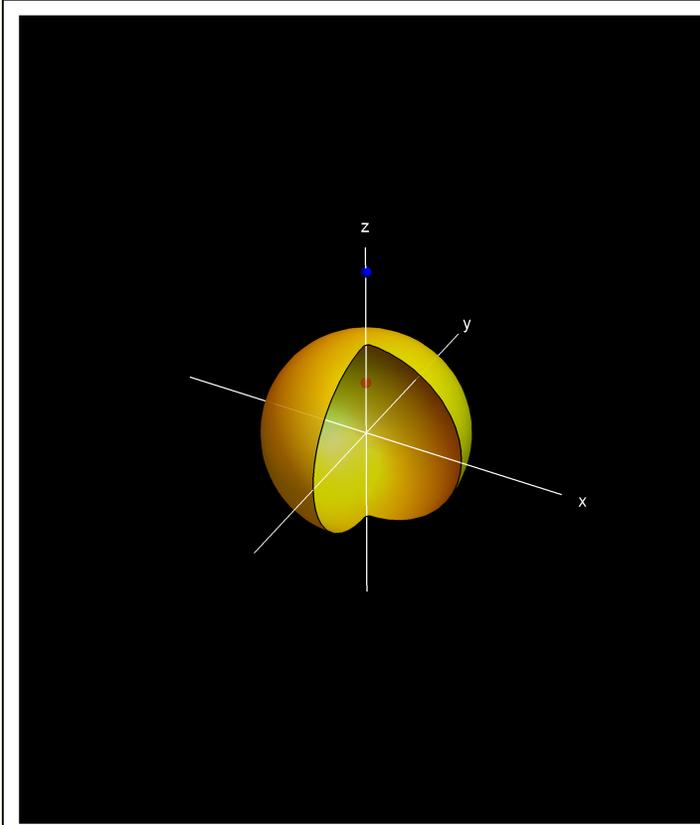
Purpose

This is the first of four notebooks dealing with electrostatic topics and calculations as treated in Jackson *Classical Electrodynamics* — my graduate textbook some fifty years ago. I recently picked up my heavily marked and taped copy and thought it would be fun to review some of the material that caused me difficulty as a student. That text is so rich in material that I had to pick and choose, limiting myself to examples involving spherical symmetry.

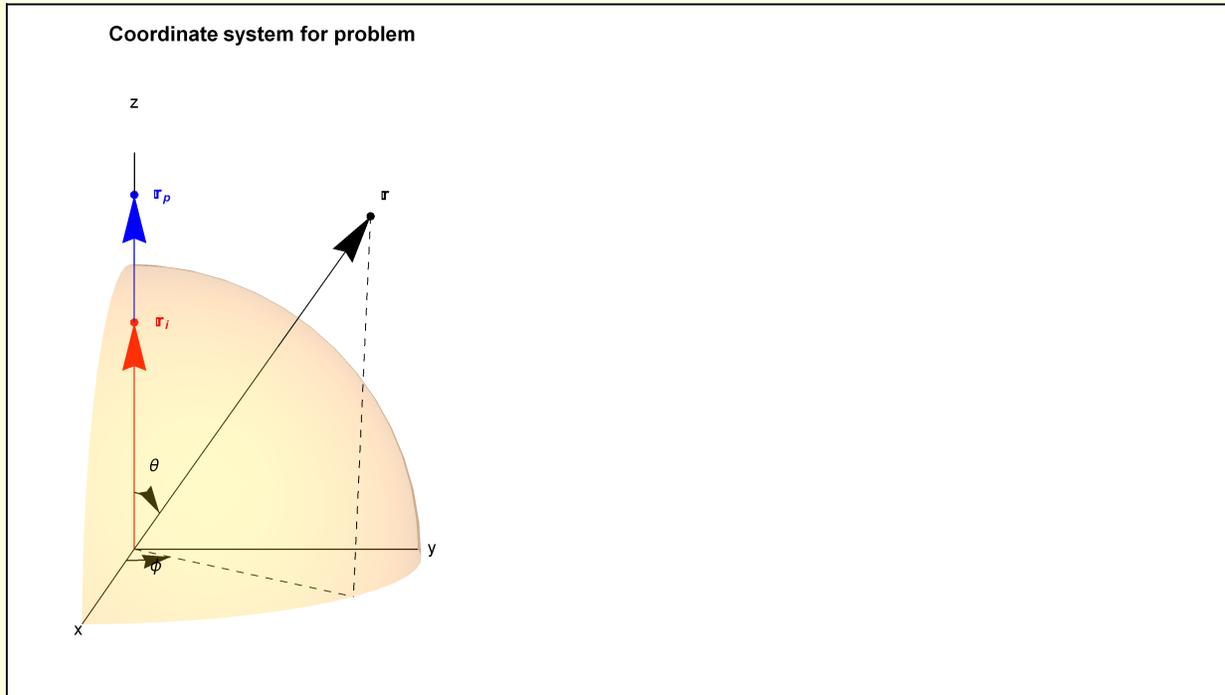
A second motivation of this work was to examine how much of the calculation process I could accomplish with Mathematica. The operations involve manipulation of coordinate-free vector expressions, use of sometimes confusing decorated notations, and occasional tedious sub calculations. It is also useful to have visualizations that elucidate various aspects of the calculation.

This notebook focuses on the classic problem of determining the potential associated with a point charge outside a grounded spherical conductor, a calculation appearing in most textbooks. In this notebook the problem is solved using the method of images. In subsequent notebooks, this same problem will be treated in other ways.

I Potential for point charge outside a grounded conducting shell



The basic question is whether a set of one or more “image” charges can be placed so that the potential on the surface of the conducting sphere is zero. For cases involving a high degree of symmetry the answer is often yes. In the figure above, the point charge is chosen, without loss of generality, to lie on the positive on the positive z-axis (the blue point in the figure). Elementary symmetry considerations convince that the image charge also must lie on the z-axis and inside the conducting sphere (the red point) if it is to satisfy the boundary condition $\Phi = 0$ on the surface of the conductor.



The potential due to the charge q_p and its image q_i is

$$\Phi(\mathbf{r}) = \frac{q_p}{|\mathbf{r} - \mathbf{r}_p|} + \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \quad (1)$$

where \mathbf{r} is the vector to the observation point, \mathbf{r}_p is the location of charge p , etc.

Based on my personal background, I choose cgs units rather than the more modern S.I.
https://en.wikipedia.org/wiki/Gaussian_units#Major_differences_between_Gaussian_and_SI_units

I will want to manipulate vector quantities as symbols. I choose to represent vectors as “doublestruck” characters. I will also “symbolize” various subscripted variables so they can be more easily used in calculations (this is a Mathematica related peculiarity - subscripted variables have an underlying structure that sometimes requires special handling).

In[3]:=

<< Notation`

In[4]:=

```
Symbolize[  $\mathbf{r}_p$  ]; Symbolize[  $\mathbf{r}_i$  ]; Symbolize[  $q_p$  ]; Symbolize[  $q_i$  ];
Symbolize[  $r_p$  ]; Symbolize[  $r_i$  ];
Symbolize[  $\mathbf{1}_r$  ];
Symbolize[  $\mathbf{1}_z$  ];
```

Calculation of the location and magnitude of the image charge:

$$w1[1] = \Phi[r] == \frac{q_p}{\text{Norm}[r - r_p]} + \frac{q_i}{\text{Norm}[r - r_i]}$$

$$\Phi[r] == \frac{q_i}{\text{Norm}[r - r_i]} + \frac{q_p}{\text{Norm}[r - r_p]}$$

Introduce unit vectors

$$w1[2] = w1[1][[1]] == (w1[1][[2]] /. \{r \to r \mathbf{1}_r, r_p \to r_p \mathbf{1}_z, r_i \to r_i \mathbf{1}_z\})$$

$$\Phi[r] == \frac{q_i}{\text{Norm}[\mathbf{1}_r r - \mathbf{1}_z r_i]} + \frac{q_p}{\text{Norm}[\mathbf{1}_r r - \mathbf{1}_z r_p]}$$

The sphere, with radius a, is grounded

$$w1[3] = w1[2] /. r \to a /. \Phi[r] \to \theta$$

$$\theta == \frac{q_i}{\text{Norm}[a \mathbf{1}_r - \mathbf{1}_z r_i]} + \frac{q_p}{\text{Norm}[a \mathbf{1}_r - \mathbf{1}_z r_p]}$$

To facilitate the valuation of this expression, I introduce a “dot product” operator. Doing so is hardly worth the effort in this case, but there are situations where it is useful to define such operators that can facilitate calculations.

```
Clear[dot]; SetAttributes[dot, Orderless];
dot[a_ + b_, c_ + d_] := dot[a, a] + dot[a, b] + dot[b, a] + dot[b, b];
dot[a_ b_, c_] /; And[FreeQ[a, 1_r], FreeQ[a, 1_z]] := a dot[b, c];
dot[1_r, 1_r] := 1; dot[1_z, 1_z] := 1;
dot[1_r, 1_z] := Cos[theta];
```

With these definitions, a good result is generated as soon as Norm is defined in terms of dot.

$$w1[4] = w1[3] /. \text{Norm}[a_] \to \sqrt{\text{dot}[a, a]}$$

$$\theta == \frac{q_i}{\sqrt{a^2 + r_i^2 - 2 a r_i \text{Cos}[\theta]}} + \frac{q_p}{\sqrt{a^2 + r_p^2 - 2 a r_p \text{Cos}[\theta]}}$$

This equation for the image charge parameters must be true for all θ . Two specific equations are

$$w1[5] = \{w1[4] /. \theta \to \theta, w1[4] /. \theta \to \pi\}$$

$$\left\{ \theta == \frac{q_i}{\sqrt{a^2 - 2 a r_i + r_i^2}} + \frac{q_p}{\sqrt{a^2 - 2 a r_p + r_p^2}}, \theta == \frac{q_i}{\sqrt{a^2 + 2 a r_i + r_i^2}} + \frac{q_p}{\sqrt{a^2 + 2 a r_p + r_p^2}} \right\}$$

```
w1[6] = Solve[w1[5], {q_i, r_i}] // Simplify[#, Assumptions -> {a > 0, r_p > 0}] &
{{q_i -> -q_p, r_i -> r_p}, {q_i -> -\frac{a q_p}{r_p}, r_i -> \frac{a^2}{r_p}}}
```

The physical solutions are

```
w1[7] = w1[6][[2]]
{q_i -> -\frac{a q_p}{r_p}, r_i -> \frac{a^2}{r_p}}
```

Then, the potential is

```
w1[11] = w1[2] /. Norm[a_] -> Sqrt[Dot[a, a]] /. w1[7] /. Phi[1, r] -> Phi[r, theta]
Phi[r] == -\frac{a q_p}{r_p \sqrt{r^2 + \frac{a^4}{r_p^2} - \frac{2 a^2 r \cos[\theta]}{r_p}}} + \frac{q_p}{\sqrt{r^2 + r_p^2 - 2 r r_p \cos[\theta]}}
```

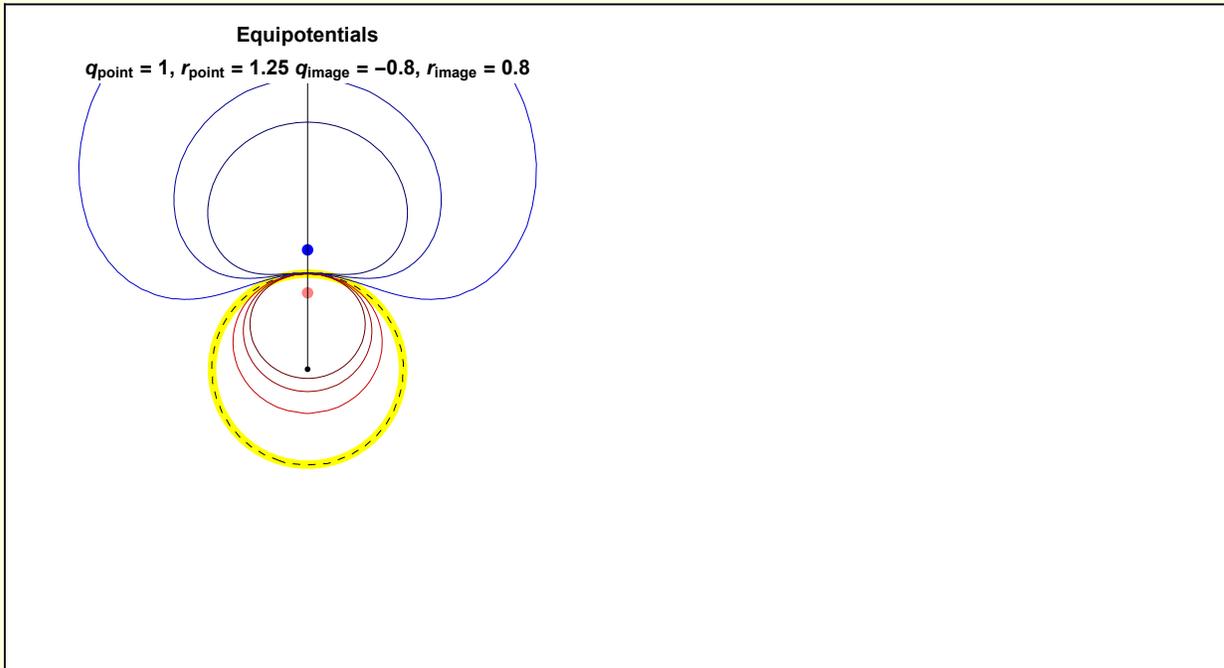
I plot Φ in the x-z plane

```
w1[12] = Phi[x, y, z] ==
TransformedField["Spherical" -> "Cartesian", w1[11][[2]], {r, theta, phi} -> {x, y, z}]
Phi[x, y, z] == -\left(\left(q_p \left(-r_p \sqrt{\left(\frac{a^4}{r_p^2} + x^2 + y^2 - \frac{2 a^2 z}{r_p} + z^2\right)} + a \sqrt{r_p^2 + x^2 + y^2 - 2 r_p z + z^2}\right)\right) / \left(r_p \sqrt{\left(\frac{a^4}{r_p^2} + x^2 + y^2 - \frac{2 a^2 z}{r_p} + z^2\right)} \sqrt{r_p^2 + x^2 + y^2 - 2 r_p z + z^2}\right)\right)
```

```
w1[12] = MapEqn[Simplify[#, {r_p > 0}] &, w1[12]] /. y -> 0
```

```
Phi[x, 0, z] == q_p \left( \frac{1}{\sqrt{r_p^2 + x^2 - 2 r_p z + z^2}} - a / (\sqrt{a^4 - 2 a^2 r_p z + r_p^2 (x^2 + z^2)}) \right)
```

Some equipotential lines for a representative numerical example (plot is generated in Appendix A)



Blue contours indicate positive potential, red contours indicate negative potential. At the surface of the conductor (yellow circle), the equipotential is zero (black dashed line).

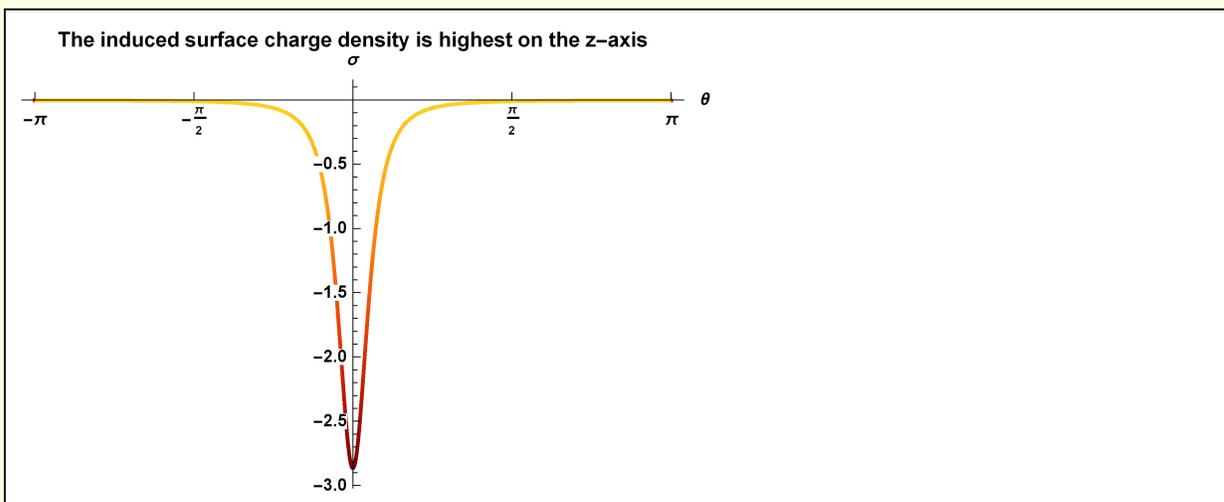
Various quantities of physical interest can be calculated from the potential

Calculation of surface charge density:

$$w1[13] = \sigma[a, \theta] == -\frac{1}{4\pi} D[w1[11][[2]], r] /. r \to a //$$

Simplify[#, Assumptions $\to \{a > 0, r_p > 0, r_i > 0\}$] &

$$\sigma[a, \theta] == (q_p (a^2 - r_p^2)) / (4 a \pi (a^2 + r_p^2 - 2 a r_p \cos[\theta])^{3/2})$$

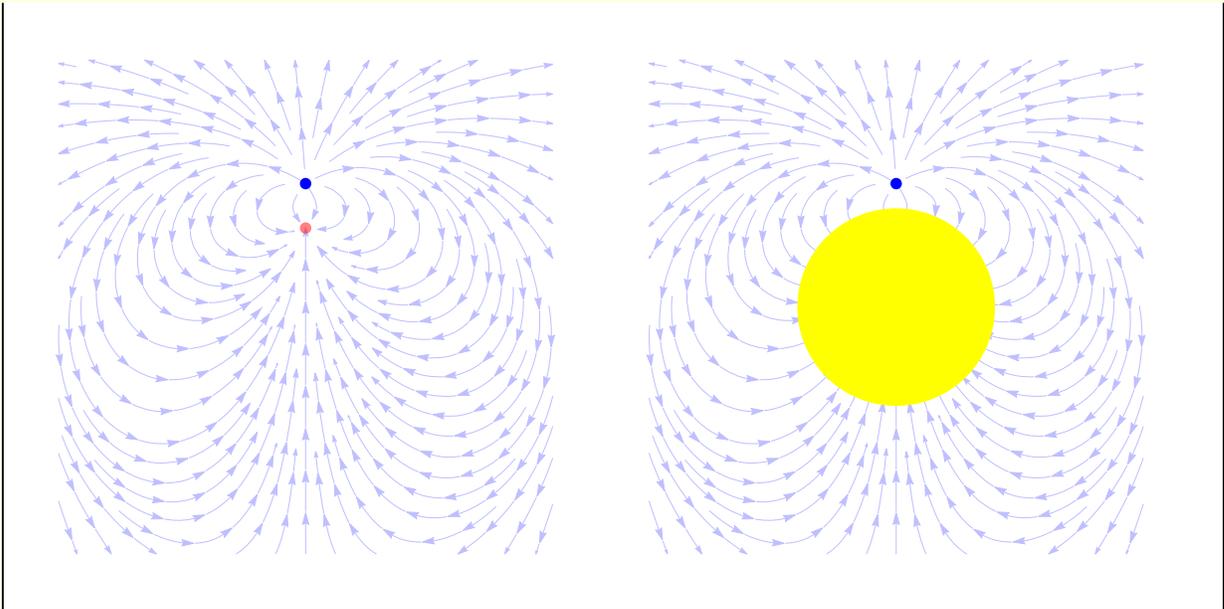


Electric field:

```
w1[14] = E[x, z] == -Grad[w1[12][[2]], {x, z}]
```

$$\mathbb{E}[x, z] = \left\{ -q_p \left(-\frac{x}{(r_p^2 + x^2 - 2r_p z + z^2)^{3/2}} + \frac{a r_p^2 x}{(a^4 - 2a^2 r_p z + r_p^2 (x^2 + z^2))^{3/2}} \right), \right. \\ \left. -q_p \left(-\frac{(-2r_p + 2z)}{(2(r_p^2 + x^2 - 2r_p z + z^2))^{3/2}} + \frac{a(-2a^2 r_p + 2r_p^2 z)}{(2(a^4 - 2a^2 r_p z + r_p^2 (x^2 + z^2))^{3/2}} \right) \right\}$$

Two visualizations of the electric field



Appendix A: Visualization

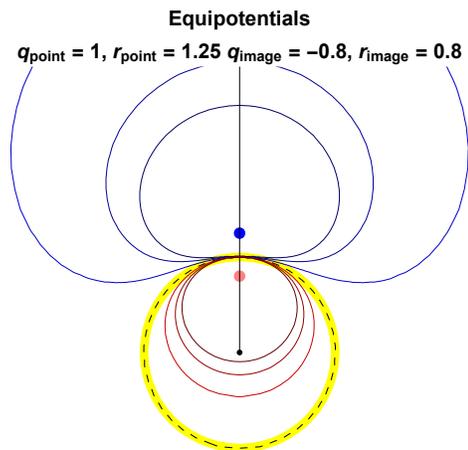
The electrostatic potential for a collection of charges can be calculated using the function

In[7]:=

```
Clear[ElectroStaticPotential];
ElectroStaticPotential[q_List, r_q_List, r_] := Sum[
  
$$\frac{q[[i]]}{\text{Norm}[r - r_q[[i]]]}$$
, {i, Length[q]}]
```

Equipotentials for representative parameters

```
Module[{a = 1, qp = 1, rp = 1.25, qi, ri, xp, xi, xmax = 3},
  qi = -qp  $\frac{a}{r_p}$ ; ri =  $\frac{a^2}{r_p}$ ;
  xp = {0, rp}; xi = {0, ri};
  ShowConductorAndCharges[a, qp, qi, xp, xi, xmax]
```



In[9]:=

```
Clear[ShowConductorAndCharges];
ShowConductorAndCharges[a_, qp : _, qi : _, xp : _, xi : _, zMax_] :=
Module[{conductor, zAxis, positiveCharge, negativeCharge, lab, G},

  conductor = {Yellow, Thickness[0.015], Circle[{0, 0}, 1]};
  positiveCharge = {Blue, PointSize[0.02], Point[xp]};
  negativeCharge = {Opacity[0.5], Red, PointSize[0.02], Point[xi]};
  zAxis = {Black, PointSize[0.01], Point[{0, 0}], Line[{{0, 0}, {0, zMax}}]};
  lab =
  Stl@StringForm["Equipotentials\nqpoint = ``, rpoint = `` qimage = `` , rimage = ``",
    qp, xp[[2]], qi, xi[[2]]];
  G[1] = Graphics[{conductor, positiveCharge, negativeCharge, zAxis},
    PlotRange → {{-zMax, zMax}, {-zMax, zMax}}, ImageSize → 300, PlotLabel → lab];

  G[2] = ContourPlot[ElectroStaticPotential[{qp, qi}, {xp, xi}, {x, y}],
    {x, -zMax, zMax}, {y, -zMax, zMax}, ClippingStyle → Automatic,
    Contours → {-0.15, -0.1, -0.05, 0, 0.1, 0.2, 0.3}, ContourStyle →
    {Darker[Red, 0.6], Darker[Red, 0.4], Darker[Red, 0.2], Directive[Black, Dashed],
    Darker[Blue, 0.2], Darker[Blue, 0.4], Darker[Blue, 0.6]},
    ContourShading → None, ImageSize → 300, PlotLabel → lab];

  Show[G[1], G[2]]]
```

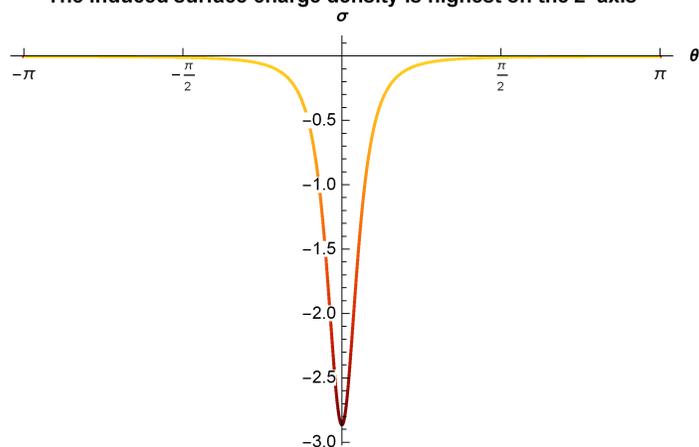
Function for the induced charge density

In[11]:=

```
Clear[InducedChargeDensity];
InducedChargeDensity[θ_, a_, q_p : _, r_p : _, q_i : _, r_i : _] :=
  (q_p (a^2 - r_p^2)) / (4 a π (a^2 + r_p^2 - 2 a r_p Cos[θ])^(3/2))
```

```
Module[{a = 1, q_p = 1, r_p = 1.25, xMax = 3, q_i, r_i, lab},
  q_i = -q_p a / r_p; r_i = a^2 / r_p;
  lab = Stl["The induced surface charge density is highest on the z-axis"];
  Plot[InducedChargeDensity[θ, a, q_p, r_p, q_i, r_i], {θ, -π, π},
    ColorFunction -> "SolarColors", PlotRange -> All, AxesLabel -> {Stl["θ"], Stl["σ"]},
    AxesOrigin -> {0, 0}, Ticks -> {{-π, -π/2, 0, π/2, π}, Automatic}, PlotLabel -> lab]]
```

The induced surface charge density is highest on the z-axis



Function for electric field

In[13]:=

```
Clear[ElectricField];
ElectricField[x_, z_, a_, q_p : _, r_p : _, q_i : _, r_i : _] :=
  {-q_p (-x / (r_p^2 + x^2 - 2 r_p z + z^2)^(3/2)) + (a r_p^2 x) / (a^4 - 2 a^2 r_p z + r_p^2 (x^2 + z^2))^(3/2),
  -q_p (-((-2 r_p + 2 z) / (2 (r_p^2 + x^2 - 2 r_p z + z^2)^(3/2))) +
  (a (-2 a^2 r_p + 2 r_p^2 z)) / (2 (a^4 - 2 a^2 r_p z + r_p^2 (x^2 + z^2))^(3/2))}
```

```

Clear[ShowElectricField];
ShowElectricField[a_, qp : _, rp : _, qi : _, ri : _, zMax_] :=
Module[{conductor, positiveCharge, negativeCharge, zAxis, colordata, range, G},
range → {{-zMax, zMax}, {-zMax, zMax}};
colordata = ColorData["SolarColors"];

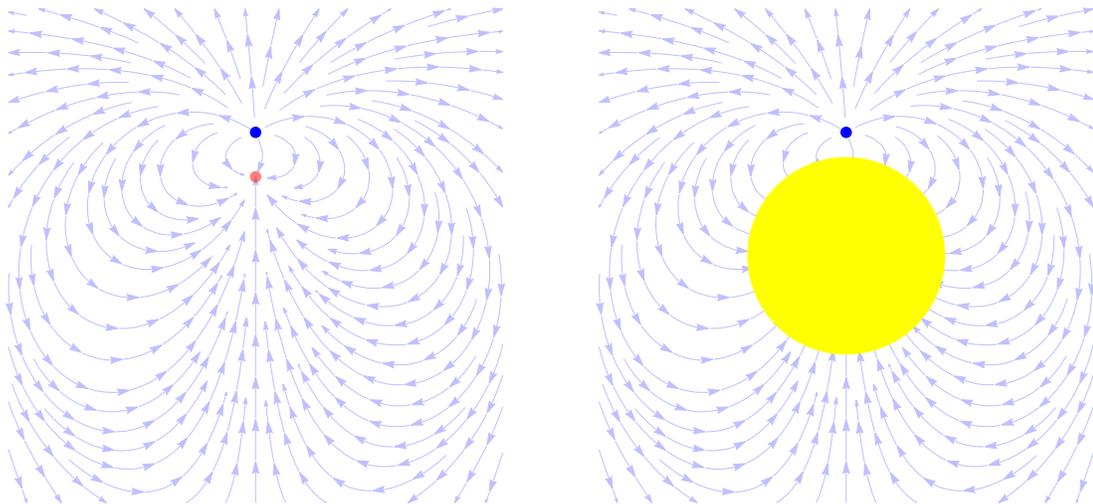
conductor = {Yellow, Thickness[0.015], Disk[{0, 0}, 1]};
positiveCharge = {Blue, PointSize[0.02], Point[{0, rp}]};
negativeCharge = {Opacity[0.5], Red, PointSize[0.02], Point[{0, ri}]};
G[1] = Graphics[{conductor, positiveCharge},
PlotRange → range, ImageSize → 300, Frame → None, AxesLabel → None];

G[2] = StreamPlot[ElectricField[x, z, a, qp, rp, qi, ri],
{x, -zMax, zMax}, {z, -zMax, zMax}, ImageSize → 300,
StreamStyle → Lighter[Blue, 0.75], Frame → None, AxesLabel → None];

G[3] = Show[G[2], G[1]];
G[4] = Show[G[2], Graphics[{positiveCharge, negativeCharge},
PlotRange → range, ImageSize → 300, Frame → None, AxesLabel → None]];
Grid[{{G[4], G[3]}}];

Module[{a = 1, qp = 1, rp = 1.25, xMax = 2.5, qi, ri},
qi = -qp  $\frac{a}{r_p}$ ; ri =  $\frac{a^2}{r_p}$ ;
ShowElectricField[a, qp, rp, qi, ri, xMax]]

```



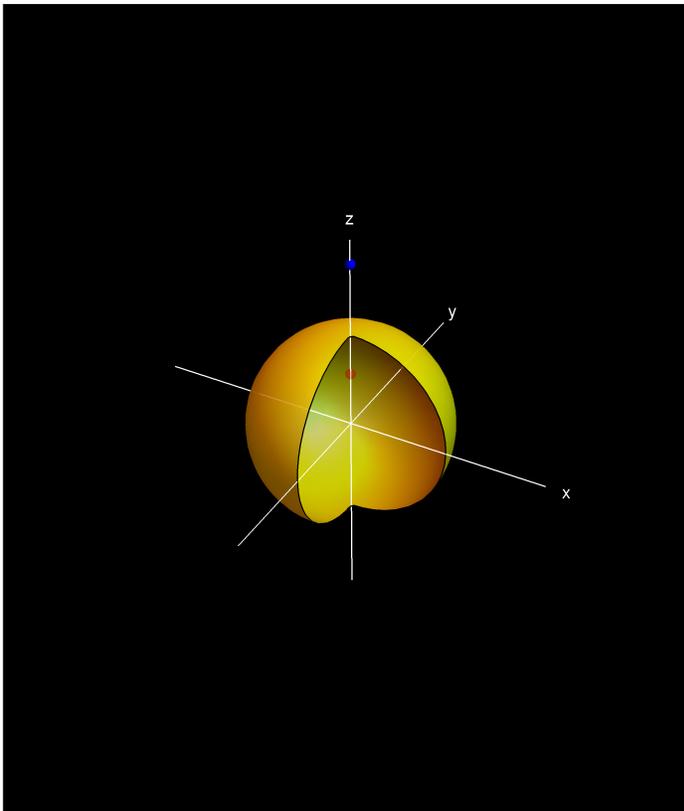
Visualization of the conducting sphere and associated charges

```

Module[{a = 1, xMax = 2, zCharge = 1.75,
  rCharge = 0.05, zImageCharge, ptCharge, ptImageCharge, G},
  ptCharge = {Blue, Sphere[{0, 0, zCharge}, rCharge]};
  zImageCharge = a2/zCharge;
  ptImageCharge = {Red, Opacity[0.25], Sphere[{0, 0, zImageCharge}, rCharge]};

  G[1] = Graphics3D[
    {White, Line[{{-xMax, 0, 0}, {xMax, 0, 0}}], Line[{{0, -xMax, 0}, {0, xMax, 0}}],
    Line[{{0, 0, -xMax}, {0, 0, xMax}}], Text["x", 1.1 {xMax, 0, 0}],
    Text["y", 1.1 {0, xMax, 0}], Text["z", 1.1 {0, 0, xMax}], ptCharge,
    ptImageCharge}, Boxed → False, Background → Black, PlotLabel →
    St1["Conducting sphere with external point charge\nand internal image charge"]];
  G[2] = ContourPlot3D[{x2 + y2 + z2 == a2}, {x, -1, 1}, {y, -1, 1},
    {z, -1, 1}, Mesh → False, ViewPoint → Right,
    ContourStyle → Directive[Yellow, Opacity[0.8], Specularity[White, 30]],
    BoxRatios → Automatic, Boxed → False, Background → Black,
    RegionFunction → Function[{x, y, z}, x < 0 || y > 0]};
  Show[G[1], G[2]]]

```



In[17]:=

```

Clear[ShowCoordinateSystemForImageProblem];
ShowCoordinateSystemForImageProblem[] :=
Module[{a = 1, rp = 1.25,  $\delta$  = 0.05, xMax = 1, 0 = {0, 0, 0},
  ri, axes,  $\theta$ arc,  $\phi$ arc,  $\xi$ arc, r, rp, ri, rprojection, OP, liner,
  linerp, lineri, lineBase, range3D, lab, G, StoC, PointOnArc},
StoC[r_,  $\theta$ _,  $\phi$ _] := {r Sin[ $\theta$ ] Cos[ $\phi$ ], r Sin[ $\theta$ ] Sin[ $\phi$ ], r Cos[ $\theta$ ]};
(* point on the arc connecting points P1 and P2*)
PointOnArc[ $\alpha$ _ (* $0 < \alpha < 1$ *)], P1_, P2_, rMag_ (*distance of arc from 0*)] :=
Module[{P $\alpha$  = P1 +  $\alpha$  (P2 - P1)}, rMag P $\alpha$  / Norm[P $\alpha$ ]];

range3D = {{-0.1 xMax, xMax}, {-0.1 xMax, xMax}, {-0.1 xMax, 1.4 xMax}};
lab = Stl["Coordinate system for problem"];
(* spherical octant *)
G[0] = ParametricPlot3D[{{Cos[ $\phi$ ] Sin[ $\theta$ ], Sin[ $\phi$ ] Sin[ $\theta$ ], Cos[ $\theta$ ]},
  { $\phi$ , 0,  $\pi/2$ }, { $\theta$ , 0,  $\pi/2$ }, Mesh  $\rightarrow$  False, PlotStyle  $\rightarrow$  Opacity[.25],
  Axes  $\rightarrow$  None, Boxed  $\rightarrow$  False, ViewPoint  $\rightarrow$  Right, ImagePadding  $\rightarrow$  10,
  PlotRange  $\rightarrow$  range3D, PlotLabel  $\rightarrow$  lab, ImageSize  $\rightarrow$  250];

r = StoC[1.5,  $\frac{\pi}{4}$ ,  $\frac{\pi}{4}$ ];
rprojection = StoC[1,  $\frac{\pi}{2}$ ,  $\frac{\pi}{4}$ ];
rp = {0, 0, rp};
ri = a2/rp; ri = {0, 0, ri};

axes = {{Line[{{0, 0, 0}, {1, 0, 0}], Text["x", 1.05 {1, 0, 0}]},
  {Line[{{0, 0, 0}, {0, 1, 0}], Text["y", 1.05 {0, 1, 0}]},
  {Line[{{0, 0, 0}, {0, 0, 1.5}], Text["z", 1.05 {0, 0, 1.5}]}];
liner = {Black, Arrowheads[Medium],
  Arrow[{0, r}], PointSize[0.02], Point[r], Text[Stl["r"], 1.05 r]};
linerp = {Blue, Arrowheads[Medium], Arrow[{0, rp}], PointSize[0.02],
  Point[rp], Text[Stl["rp"], rp + {0, 0.1, 0}]}];
lineri = {Red, Arrowheads[Medium], Arrow[{0, ri}], PointSize[0.02],
  Point[ri], Text[Stl["ri"], ri + {0, 0.1, 0}]}];
lineBase = {Dashed, Line[{r, rprojection}], Line[{0, rprojection}]}];
 $\theta$ arc = With[{ $\theta$ Vals = StoC[0.2, #,  $\pi/4$ ] & /@ Range[0,  $\pi/4$ ,  $\pi/64$ ]},
  {Black, Arrowheads[Small], Arrow[ $\theta$ Vals], Text[" $\theta$ ", 1.6  $\theta$ Vals[[8]]]}];
 $\phi$ arc = With[{ $\phi$ Vals = StoC[0.2,  $\pi/2$ , #] & /@ Range[0,  $\pi/4$ ,  $\pi/64$ ]},
  {Black, Arrowheads[Small], Arrow[ $\phi$ Vals], Text[" $\phi$ ", 1.6  $\phi$ Vals[[8]]]}];

G[1] = Graphics3D[{axes, { $\theta$ arc,  $\phi$ arc(*,  $\xi$ arc*)},
  {liner, lineBase, linerp, lineri}}, Boxed  $\rightarrow$  False, PlotRange  $\rightarrow$  range3D];
Show[G[0], G[1], ImageSize  $\rightarrow$  250]

```

In[19]:=

ShowCoordinateSystemForImageProblem []

Out[19]=

